Lecture 8

Proof by Contradiction, Proof By Exhaustion

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Example: Prove that "For an integer n, n is odd if and only if n^2 is odd."

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Example: Prove that "For an integer n, n is odd if and only if n^2 is odd."

1. For an integer n, if n is odd, then n^2 is odd.

- We can prove the above statement by proving the below statements:

Because,

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Proving both "if p, then q" and "if q, then p", proves "p if and only if q".

Example: Prove that "For an integer n, n is odd if and only if n^2 is odd." 1. For an integer n, if n is odd, then n^2 is odd. 2. For an integer n, if n^2 is odd, then n is odd.

- We can prove the above statement by proving the below statements:

Suppose we want to prove proposition *p* true.

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Assume *p* is false.

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 $\neg p$

Suppose we want to prove proposition *p* true.

Assume $\neg p$ is true.

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$\neg p \rightarrow q_1 \rightarrow q_2 \rightarrow q_3 \rightarrow \dots$

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Assume $\neg p$ is true.

$\neg p \rightarrow q_1 \rightarrow q_2 \rightarrow q_3 \xrightarrow{q_4} q_5 \rightarrow \dots \rightarrow r$

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Suppose we want to prove proposition p true.

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Above chain of deductions shows that $(\neg p \rightarrow r) \land (\neg p \rightarrow \neg r)$ is true.

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 $\neg p \rightarrow (r \land \neg r)$ can be true only when $\neg p$ is false. Hence p is true.

 $(p \lor \neg r) \equiv p \lor (r \land \neg r) \equiv \neg p \rightarrow (r \land \neg r)$

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Above chain of deductions shows that $\neg p \rightarrow p$ is true. But $\neg p \rightarrow p$ is true only when p is true. Hence proved.

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 $\sqrt{2} = \frac{a}{b} \tag{1}$

where $b \neq 0$, and a and b have no common factors.

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have a common factor", $\neg p$ must be false. Thus, p, i.e., $\sqrt{2}$ is irrational, must be true.



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Let p_1, p_2, \ldots, p_n denote the list of all the primes in ascending order.

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Proof: For the sake of contradiction, suppose there are only finitely many primes.

Let p_1, p_2, \ldots, p_n denote the list of all the primes in ascending order.

Consider the number $a = p_1 \cdot p_2 \cdot p_3 \cdot \dots \cdot p_n + 1$.

Clearly, $a > p_n$ and thus cannot be a prime number.

Theorem: There are infinitely many prime numbers. **Proof:** For the sake of contradiction, suppose there are only **finitely** many primes. Let $p_1, p_2, ..., p_n$ denote the list of all the primes in ascending order. Consider the number $a = p_1 . p_2 . p_3 p_n + 1$. Clearly, $a > p_n$ and thus cannot be a prime number. Since a > 1, there must exist a prime divisor of a, say p_k .

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So,

$$\frac{1}{p_k} = c - (p_1 \cdot p_2 \cdot \dots \cdot p_{k-1} p_{k+1} \cdot \dots \cdot p_k)$$

The expression on the right is an integer

Dividing both sides with p_k gives us,

$$p_1 . p_2 p_{k-1} p_{k+1} p_n + \frac{1}{p_k} = c$$

So,

$$\frac{1}{p_k} = c - (p_1 . p_2 p_{k-1} p_{k+1} . . . p_k)$$

The expression on the right is an integer, while the expression on the left is not an integer.

Dividing both sides with p_k gives us,

$$p_1 \cdot p_2 \dots \cdot p_{k-1} p_{k+1} \dots \cdot p_n + \frac{1}{p_k} = c$$

So,

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- **Theorem:** Every non-zero rational number is expressible as product of two irrational numbers.
 - If *r* is a non-zero rational number, then *r* is a product of two irrational numbers.
 - Since r is a non-zero rational number, $r = \frac{a}{b}$, where $a \neq 0$ and $b \neq 0$ are integers.

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