Lecture 8

Introduction to Proofs, Direct Proofs, Proofs by Contraposition

An Example of a Proof

Theorem: If n is an odd integer, then n^2 is also an odd integer.

Proof: By the definition of an odd integer,

$n = 2k + 1$, where k is some integer.

Square on both sides of $n = 2k + 1$ to obtain the value of n^2.

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2.(2k^2 + 2k) + 1$$

Thus, $n^2 = 2k' + 1$, where $k' = 2k^2 + 2k$. Hence, n^2 is an odd integer. ■

Using the existing facts such as (1) If $a = b$, then $a^2 = b^2$, (2) $(a+b)^2 = a^2 + b^2 + 2ab$, etc.

Using the axiom that "If $a = b$ and $b = c$, then $a = c$." without stating.

Some Important Terms

A **theorem** is an important mathematical statement that can be proved.

A **proposition** is a less important mathematical statement that can be proved.

A **lemma** is a mathematical statement that can be proved and useful in proving other results.

A **corollary** is a mathematical statement that can be established directly from a theorem that has been proved.

A **conjecture** is a mathematical statement that is being proposed to be a true statement.

Methods of Proofs

Some of the **types of informal proofs** that we will learn are:

- Direct Proofs.
- Proof by Contraposition.
- Proof by Contradiction.
- Proof by Exhaustion.
- Proof by Induction.

Direct Proofs

In **direct proofs**, we establish the truth of a mathematical statement by a **straightforward combination** of axioms, definitions, rules of inferences, and existing facts.

For instance, the proof of the last theorem was a direct proof.

Let's do one more!

Theorem: If n is an odd integer, then n is the difference of squares of two integers.

Major Tip: Before writing the proof, ensure the following:

- You understand the statement you intend to proof.
- Play with the examples and convince yourself that the statement is true.
- You have an outline of the proof.

Examples: Direct Proofs

Theorem: If n is an odd integer, then n is the difference of squares of two integers.

Rough Work: Let's see whether the statement is true for some odd numbers.

$$11 = 6^2 - 5^2, \quad 9 = 5^2 - 4^2, \quad 7 = 4^2 - 3^2, \quad 5 = 3^2 - 2^2$$

We want two number k_1 and k_2, such that $n = k_1^2 - k_2^2$.

Try to express k_1 and k_2 in term of n using above examples.

$$11 = 6^2 - 5^2$$

$$n \qquad \left(\frac{n+1}{2}\right) \qquad \left(\frac{n-1}{2}\right)$$

Verify the guess, i.e., $n = \left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2$.

Examples: Direct Proofs

Theorem: If n is an odd integer, then n is the difference of squares of two integers.

Proof: For any odd integer n, let $k_1 = \left(\frac{n+1}{2}\right)$ and $k_2 = \left(\frac{n-1}{2}\right)$.

Clearly, k_1 and k_2 are integers. ← *Easy steps do not require much justification.*

Below, we show that $n = k_1^2 - k_2^2$.

$$k_1^2 - k_2^2 = \left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2$$

$$= \frac{n^2+1+2n}{4} - \frac{n^2+1-2n}{4}$$

$$= \frac{n^2+1+2n-n^2-1+2n}{4} = \frac{4n}{4} = n$$

■

Examples: Direct Proofs

Theorem: If n is an odd integer, then n is the difference of squares of two integers.

Alternative Proof: By the definition of an odd integer,

If n is an odd integer, then $n = 2k + 1$, where k is some integer.

Below, we show that n is the difference of squares of two integers.

$$\begin{aligned} n &= 2k + 1 \\ &= 1.\,(2k + 1) \\ &= (k + 1 - k)\,.\,(k + 1 + k) \\ &= (k + 1)^2 - k^2 \quad \textit{(Using } (a+b)(a-b) = a^2 - b^2) \end{aligned}$$

$\blacksquare$

Examples: Direct Proofs

Theorem: For any positive integers a and b, if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof: ....?????..........

Direct proof doesn't seem to work here....at least not easily.

Time to introduce **Proof by Contraposition**.

Proof by Contraposition

In **proof by contraposition**, we establish the truth of mathematical statement "If p, then q" by establishing truth of logically equivalent statement "If $\neg q$, then $\neg p$".

Suppose we want to prove the below theorem.

Theorem: For any positive integers a and b, if $\underbrace{n = ab}_{p}$, then $\underbrace{a \leq \sqrt{n} \text{ or } b \leq \sqrt{n}}_{q}$.

$\neg p \ = \ n \neq ab.$

$\neg q \ = \ a > \sqrt{n}$ and $b > \sqrt{n}$. *(Apply De Morgan's law on q.)*

Equivalent statements.

Now, instead of proving the above theorem we can prove the below.

Theorem: For any positive integers a and b, if $a > \sqrt{n}$ and $b > \sqrt{n}$, then $ab \neq n$.

Examples: Proof by Contraposition

Theorem: For any positive integers a and b, if $a > \sqrt{n}$ and $b > \sqrt{n}$, then $ab \neq n$.

Proof: We will prove the contrapositive of the theorem. That is,

For any positive integers a and b, if $a > \sqrt{n}$ and $b > \sqrt{n}$, then $ab \neq n$.

We know that

$$a > \sqrt{n}$$
$$b > \sqrt{n}$$

Multiply both the inequalities

$$ab > \sqrt{n}\sqrt{n}$$
$$> n$$

Thus, $ab \neq n$.

■